International Journal of Engineering Research & Management Technology
(Peer-Reviewed, Open Access, Fully Refereed International Journal) ISSN: 2348-4039
Volume 11, Issue-5, September-October- 2024 Impact Factor: 7.09
Email: editor@ijermt.org www.ijermt.org

Numerical approaches for solving initial value problems utilizing the Modified Euler method and the Taylor series method

Rashmi Kumari, Dr. V. K. Pandey
Department of Mathematics & Radha Govind University, Ramgarh, Jharkhand
Rashmiedu86@gmail.com

1. Introduction

Numerical methods are indispensable tools for solving initial value problems (IVPs) in differential equations, especially when analytical solutions are difficult or impossible to obtain. Among these methods, the Modified Euler's method and Taylor's method stand out due to their simplicity and effectiveness. The Modified Euler's method, also known as Heun's method, is an improved version of the classical Euler's method, which reduces error by considering the average of the slopes at the beginning and the end of the interval. This approach not only enhances accuracy but also maintains the simplicity and ease of implementation that makes Euler's methods appealing for a wide range of applications [1]. On the other hand, Taylor's method leverages the Taylor series expansion to approximate solutions by incorporating higher-order derivatives of the function, thereby increasing the precision of the solution [2]. This method is particularly powerful when the derivatives of the function are known and can be computed efficiently [3]-[11]. Both methods offer unique advantages: the Modified Euler's method strikes a balance between computational efficiency and accuracy, while Taylor's method provides high precision by utilizing additional information about the function's behavior. By exploring these methods, we can gain deeper insights into the trade-offs between complexity, computational cost, and accuracy in numerical analysis. This study aims to compare and contrast these two approaches in the context of solving IVPs, highlighting their theoretical underpinnings, practical implementations, and performance in various scenarios [4]-[19]. Through detailed analysis and illustrative examples, we seek to demonstrate how these methods can be effectively applied to solve differential equations, providing valuable tools for researchers and practitioners in fields ranging from engineering to the physical sciences.

2. Preliminaries

Definitions and fundamental concepts that are used throughout this study are presented in this part.
The Riemann-Liouville (RL) and Caputo derivatives for fractional functions with interval values under the generalized Hukuhara difference. Initially, let us remember that if $x \in L^1[p,q]$ (bounded interval $[p,q]$), then the RL fractional integral $I_{p^+}^{\alpha}x$ of hierarchy $\alpha > 0$ is determined by

$\left(I_{a^+}^{\alpha}x\right)(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}\,x(s)ds$, for $t \geq a$.

The RL derivative of hierarchy $\alpha \in (0,1]$ for a real function $x \epsilon L^1[p,q]$ is characterized by $\left(D_{p+}^{\alpha} x\right)(t) = \frac{d}{dt} I_{p+}^{1-\alpha} x(t)$, that is,

$$\left(D_{p+}^{\alpha} x\right)(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_p^t (t-s)^{-\alpha} x(s) ds. \quad (1)$$

If $x \epsilon L^1[p,q]$ is a real function such that $D_{p+}^{\alpha} x$ exists on $[p,q]$, then the Caputo fractional derivative (CFD) ${}^{C}D_{p+}^{\alpha} x$ of hierarchy $\alpha \epsilon (0,1)$ is determined by

$$\left({}^{C}D_{p+}^{\alpha} x\right)(t) := \left(D_{p+}^{\alpha}[x(\cdot) - x(p)]\right)(t). \quad (2)$$

If $x \in AC[p,q]$, Next, we have that

$$\left({}^{C}D_{p+}^{\alpha} x\right)(t) = \left(I_{p+}^{1-\alpha} \frac{d}{dt} x\right)(t) = \frac{1}{\Gamma(1-\alpha)} \int_p^t (t-s)^{-\alpha} \frac{d}{ds} x(s) ds, \quad (3)$$

And

$$\left(D_{p+}^{\alpha} x\right)(t) = \left({}^{C}D_{p+}^{\alpha} x\right)(t) + \frac{(t-p)^{-\alpha}}{\Gamma(1-\alpha)} x(p). \quad (4)$$

Additionally, we have

$$\left(I_{p+}^{\alpha} D_{p+}^{\alpha} x\right)(t) = x(t) - x(p), \quad t \in [p,q] \quad (5)$$

Many definitions and studies of fractional derivatives have been proposed in the literature. Probably this is due to the fact that no harmonious definition preserves all properties of the classical integer-order derivative [5]. These definitions include the Grunwald–Letnikov, Riemann–Liouville, Weyl, Riesz and Caputo versions. However, in the Caputo case, the derivative of a constant function is zero and one can properly define the initial conditions for the fractional differential equations which can be handled by using an analogy with the classical integer-order case. For these reasons, we adopt the Caputo fractional derivative definition in this work [6].

Definition 1: A real function $u(t), t > 0$ is in the space $C_{\lambda \in R}^1$ if there exists a real number a > λ such that $u(t) = t^a v(t)$ where $v(t) \in C[0, \infty 0$, and it is in the space $C_{\lambda}^n if u^{(n)} \in C_{\lambda}^1, n \in N$.

Definition 2: The Riemann–Liouville fractional integral operator of order α > 0 associates with a real function $u(t) \in C\lambda \geq -1^1$ is defined $J_t^{\alpha}[u(t)] = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} u(\tau) d\tau$, and J_t^0 is an identity operator.

Definition 3: The Caputo time-fractional derivative of order $\alpha > 0 of\, u(t) \in C_{-1}^n, n \in N$ is defined as $D_t^{\alpha}[u(t)] = J_t^{n-\alpha}[u^{(n)}(t)] if\ n-1 < \alpha < n$ and $D_t^{\alpha}[u(t)] = u^{(n)}(t) if\ \alpha = n$. Similarly, for n being the smallest integer that exceeds α, the Caputo time-fractional derivative operator of order α is given as $D_t^{\alpha}[u(\bar{x},t)] = J_t^{n-\alpha}\left[\frac{\partial^n u(\bar{x},t)}{\partial t^n}\right] if n-1 < \alpha < n\, and\, D_t^{\alpha}[u(\bar{x},t)] = \frac{\partial^n u(\bar{x},t)}{\partial t^n} if\ \alpha = n$.

Remark 1 A direct implementation of the Caputo derivative yields $D_t^{\alpha}[t^p] = \frac{\Gamma(p+1)}{\Gamma(p-\alpha+1)} t^{p-\alpha}$ for $p > 0$ and $D_t^{\alpha}[c] = 0$ where c is a constant. Also, it is easy to see that the Caputo derivative is a left inverse of the

Riemann–Liouville integral but not a right inverse. Specifically, for $n-1<\alpha\le n, n\in N$ and $u(t)\in C^n_{\lambda\ge -1}$ we have $J_t^{\alpha}D_t^{\alpha}[u(t)]=u(t)-\sum_{k=0}^{n-1}u^k(0^+)\frac{t^k}{k!}$, where t > 0.

It should be noted here that it suffices to consider the Caputo fractional derivative of order $0 < \alpha \le 1$ since $D_t^{\alpha-(n-1)}[u^{(n-1)}(t)]$ for arbitrary order $n-1< \alpha \le n$, where $\alpha - (n - 1) \in (0, 1]$.

Definition 4: A fractional power series (FPS) expansion is an infinite series about $t = t_0$ of the form $\sum_{k=0}^{\infty}c_k(t-t_0)^{k\alpha}$ where $0\le n-1<\alpha\le n, t\ge t_0$.

Theorem 5: Suppose that u(t) has a FPS expansion about t0 as above for $t_0\le t\le t_0+R$. If the $D_t^{k\alpha}[u(t)]$ are continuous on (t_0,t_0+R) for $k \in N^*$, then $k\in N^*$ where $D_t^{k\alpha}$ is the k-fold Caputo derivative and R is the radius of convergence.

Definition: A power series of the form

$$\sum_{k=0}^{\infty} f_k(\bar{x})t^{k\alpha} \tag{6}$$

where $(\bar{x})\in I=I_1\times\ldots\times I_m\subset R^m, 0<\alpha\le 1$ and $t \ge 0$ is called a multi-fractional power series about t = 0.

Theorem 6. Suppose that $u(\bar{x},t)$ has a multi-fractional power series representation about t = 0 as above for $\bar{x}\in I$ and $0\le t\le R$. If $D_t^{k\alpha}[u(\bar{x},t)]$ are continuous on $I\times(0,R)$ for each $k \in N^*$, then $f_k(\bar{x})=\frac{D^{k\alpha}[u(\bar{x},0)]}{\Gamma(k\alpha+1)}$ where R is the radius of convergence.

3. Analytic solution of homogeneous time-invariant fractional IVP

As our approach depends mainly on constructing an analytical solution of the time fractional differential equation under consideration, we first present, in a similar fashion to the classical power series, some essential convergence theorems pertaining to our proposed solution.

Theorem 7 Let $\{f_k(\bar{x})\}_{k=0}$ be a sequence of functions $f_k: I\to R.if$ is convergent for some $t = t_0 > 0$, then it is convergent for all $t\in(0,t_0)$.

Proof Assume that convergent for $t = t_0 > 0$. Then, for fixed $\in_0>0$, there exists $N \in N$ such that $\left|f_k(\bar{x})t_0^{\alpha k}\right|<\in_0$. It follows that if $k \ge N$, we have $\left|f_k(\bar{x})t_0^{\alpha k}\right|<\in_0\left(\frac{t}{t_0}\right)^{\alpha k}$ for all $\bar{x}\in I$ and $t\in(0,t_0)$, which shows that $\sum_{k=0}^{\infty}f_k(\bar{x})t^{k\alpha}$ is absolutely convergent (and so convergent).

We remark here that if $f_0(\bar{x})$ is a bounded function on I, then the convergence at some $t = t_0 > 0$ implies the convergence on $[0,t_0)$.

Corollary 7: Let $\{f_k(\bar{x})\}_{k=0}$ be a sequence of functions $f_k: I\to R$ is divergent for some $t = t_0 > 0$, then it is divergent for all $t > t_0$.

Proof Suppose not. That is convergent for some $t > t_0$. it converges on (0,t) and thus converges at t_0, which is a contradiction

Corollary 8: Let $\{f_k(\bar{x})\}_{k=0}$ be a sequence of functions $f_k : I \to R$. Then one of the following cases is true:

p1: The series converges only at t = 0;

p2: the series converges for all $t \geq 0$;

p3: there exists R > 0 (called the radius of convergence) such that converges for all $t \in (0,R)$ and diverges for all t > R.

Proof: Suppose it is the case that both p_1 and p_2 are not valid. Then there exist $\alpha,\beta \in R^+$ such that (2) converges at $t = \alpha$ and diverges at $t = \beta$. Therefore, the set $T = \{t > 0 : \sum_{k=0}^{\alpha} f_k(\bar{x})t^{k\alpha} converges\}$ is nonempty and $T \subseteq (0,\alpha)$ by. Thus R := supT exists. Now, if t > R, then divergent and if $0 < t < R$, then, by the definition of the supremum there exists $t < t_0 \in T$ such that convergent at t_0 and so by convergent on $(0,t_0)$. The other cases can be handled easily.

Now, consider the following general homogeneous time-invariant fractional initial value problem:

$$D_t^{\alpha}[u(\bar{x},t)] = F(u(\bar{x},t)), \qquad 0 \leq t < R \tag{7}$$

$$u(\bar{x},0) = f(\bar{x}) \tag{8}$$

where $D\alpha$ t is the Caputo fractional operator with $\alpha \in (0, 1]$, $u(\bar{x},t)$ is an unknown function, F is an analytic differential operator in the variables $\bar{x} = (x_1,....,x_m)$ that involves both linear and nonlinear terms, $R \in R$, and $f(x) \in C^{\infty}(R^m)$. In our next theorem, we exhibit a parallel scheme of the Taylor series method to solve problem. The method gives an analytical solution in the form of convergent multi-fractional power series without the need for linearization, perturbation, or discretization of the variables [7]. Instead of equating terms with the same degree of homogeneity, our approach depends recursively on time fractional differentiation to obtain the unknown series coefficients.

Notation We denote the coefficient extraction operator for a multi-fractional power series $G(\bar{x},t)$, which extracts a constant multiple of the coefficient of $t^{n\alpha}$ $in G, by [t^{n\alpha}]G$.

More precisely, for $n \geq 1$

$$[t^{n\alpha}]_G = [t^{n\alpha}]\sum_{k=0}^{\infty} gk(\bar{x})t^{k\alpha} = \Gamma(n\alpha+1)g_n(\bar{x}) \tag{9}$$

Note that, for a multi-fractional power series representation, $G(\bar{x},t) = \sum_{k=0}^{\infty} g_k(\bar{x})t^{k\alpha}$, we have

$$D_t^{n\alpha}[G(\bar{x},t)]_{t=0} = \Gamma(n\alpha+1)g_n(\bar{x}) = [t^{n\alpha}]G \tag{10}$$

Here $D_t^{n\alpha} = D_t^{\alpha}.....D_t^{\alpha}$ (n times).

4. Taylor's method

Taylor's method is a powerful and widely used technique for solving ordinary differential equations (ODEs) by approximating the solution through the Taylor series expansion. At its core, this method utilizes the fact that any sufficiently smooth function can be expressed as an infinite sum of its derivatives at a single

point, multiplied by the corresponding powers of the distance from that point. In practical applications, only a finite number of terms are used, transforming the infinite series into a polynomial approximation. The fundamental idea is to predict the value of the function at a future point based on its value and the values of its derivatives at the current point. This is particularly useful in numerical analysis where the exact analytical solution of an ODE is challenging or impossible to obtain.

The accuracy of Taylor's method is directly related to the number of terms used in the series; more terms typically yield a higher accuracy, but also require the computation of higher-order derivatives, which can be computationally intensive. The method is especially advantageous for problems where the derivatives can be easily calculated and where high precision is required over a short interval. In practice, Taylor's method is often implemented in conjunction with other numerical methods to improve stability and efficiency. For example, it can be combined with Runge-Kutta methods to enhance accuracy while controlling computational costs. Furthermore, Taylor's method finds applications in various fields, such as physics for modeling dynamic systems, engineering for simulating control processes, and finance for predicting the behavior of complex financial instruments. The flexibility and precision of Taylor's method make it a cornerstone in the numerical solution of differential equations, enabling researchers and practitioners to tackle complex problems that are otherwise intractable with analytical methods alone.

In practical applications, the Taylor series method typically involves truncating the series after a finite number of terms, which introduces an approximation error. The order of the method is determined by the number of terms retained in the series: a higher-order Taylor method includes more terms and generally yields more accurate results, albeit at the cost of increased computational complexity. Specifically, for an ODE of the form $y'(t)=f(t,y(t))y'(t) = f(t, y(t))y'(t)=f(t,y(t))$ with initial condition $y(t_0)=y_0$.

One of the significant advantages of Taylor's method is its potential for high accuracy, especially when higher-order derivatives are included. However, this also poses a challenge because calculating higher-order derivatives can be analytically complex and computationally intensive. Moreover, the method's efficiency and accuracy depend heavily on the choice of step size. Reduces truncation error but increases the number of steps required, whereas a larger can lead to significant approximation errors.

Taylor's method is particularly beneficial for problems where the solution is expected to be smooth and where high accuracy is paramount. However, its practical use is often limited by the difficulty of obtaining higher-order derivatives and the computational burden associated with evaluating these terms. In many real-world applications, Taylor's method is complemented or replaced by other numerical techniques, such as Runge-Kutta methods, which offer a balance between accuracy and computational efficiency without the need for higher-order derivatives.

The fractional differential equation problems have been successfully approximated with correct results by the Taylor series method (TSM), whose foundation is the Taylor series. All of the coefficients in the TS of the solution may be obtained in TSM by solving the recurrence equations that are derived from the supplied DE. Fractional derivative is used to generalize the traditional TSM, which is the basis for the fractional differential transform method (FDTM). Presented the generalized differential transform method (GDTM) makes use of the generalized Taylor formula. To determine the generalized TS coefficients with complicated nonlinear functions, this work aims to obtain efficient techniques.

Theorem: 2

Consider that $(D_t^{\alpha})^k f(t) \epsilon C(0,q]$ for $k = 0,1,\dots,n+1$, where$0 < \alpha \leq 1$; then one has

$$f(t) = \sum_{i=0}^{n} \frac{t^{j\alpha}}{\Gamma(i\alpha+1)} \left((D_t^{\alpha})^i f\right)(0) + \frac{\left((D_t^{\alpha})^{n+1} f\right)(\eta)}{\Gamma((n+1)\alpha+1)} t^{(n+1)\alpha} \quad (11)$$

For all $t \in (o,q]$ with $0 \leq \eta \leq t$.

The GDT of the kth derivative for an analytic function $f(t)$can be summed up as follows:

$$F(k) = \frac{1}{\Gamma(\alpha k+1)} [(D_t^{\alpha})^k f(t)]_{t=0}, \quad (12)$$

Where $0 < \alpha \leq 1, k = 0,1,2,\dots,$

The generalized differential inverse transform of $F(k)$ is described as follows:

$$f(t) = \sum_{k=0}^{\infty} F(k) t^{\alpha k} . \quad (13)$$

Theorem: 3 Suppose that $F(K), G(k)$, and $H(k) f(t), g(t)$, and $h(t)$, respectively. Then, the attributes listed below are met.

If $f(t) = g(t) \pm h(t), then\ F(k) = G(k) \pm H(k).$

i. *If* $f(t) = pg(t), where\ p\ is\ a\ constant, then\ F(K) = pG(k).$

ii. *If* $f(t) = g(t)h(t), then\ F(k) = \sum_{l=1}^{k} G(l)H(k-1).$

iii. *If* $f(t) = D_t^{\alpha} g(t), then\ F(k) = (\Gamma(\alpha(k+1)+1)/\Gamma(\alpha k+1)G(k+1).$

It is evident that the demonstration of these statements for the Caputo derivative equally obviously holds for the modified RL derivative.

4. Numerical Approximations

Numerical approximations are essential tools in various scientific and engineering disciplines, where exact solutions to mathematical problems are either impossible or impractical to obtain. These approximations allow us to estimate solutions with a high degree of accuracy, using methods that leverage computational power and advanced mathematical techniques. One common approach is the use of finite difference methods, which approximate derivatives by considering the values of functions at discrete points. This is particularly useful in solving differential equations that model physical phenomena, such as heat conduction and fluid dynamics. Another widely used method is the Monte Carlo simulation, which relies on random sampling to approximate complex integrals and probabilistic systems, often employed in financial modeling and risk assessment.

Additionally, polynomial interpolation and spline methods offer ways to estimate functions between known data points, ensuring smooth transitions and minimal error. In optimization problems, techniques like gradient descent and Newton's method iteratively approach optimal solutions by leveraging local information about the function's behavior. Furthermore, numerical integration methods, such as the trapezoidal rule and Simpson's rule, provide ways to estimate the area under curves, crucial in fields ranging from physics to economics. The convergence and stability of these numerical methods are paramount, requiring careful analysis to ensure that approximations are both accurate and reliable over time. As computational capabilities continue to advance, so too do the algorithms and techniques for numerical approximations, enabling increasingly complex and precise modeling of real-world systems. This continuous improvement highlights the importance of numerical approximations in advancing technology and understanding the intricacies of various scientific domains.

The fractional IVP under the CFD is solved numerically in several straightforward instances presented in this portion. Initially, a numerical instance is presented to establish the effectiveness of the previously recommended approaches when compared to the exact methodical solution.

Definition: 1 Let $R(t) = [\underline{R}(t), \overline{R}(t), \hat{R}(t) = [\hat{R}(t), \overline{\hat{R}}(t)]and\tilde{R}(t) = [\underline{\tilde{R}}(t), \overline{\tilde{R}}(t)]$depict the exact and approximate answers to equation (4.1). The definition of the discrepancies between the precise and approximate results is

$$E_{MFEM} = \left|\underline{R}(t) - \underline{\hat{R}}(t)\right|, \overline{E}_{MFEM} = |\overline{R}(t) - \overline{\hat{R}}(t)| \tag{14}$$

for the MFEM and

$$E_{TM} = \left|\underline{R}(t) - \underline{\tilde{R}}(t)\right|, \overline{E}_{TM} = |\overline{R}(t) - \overline{\tilde{R}}(t)| \tag{15}$$

for the modified Taylor's method.

Example: 2 Examine this fractional IVP.

$$({}^{C}\mathcal{D}_{0+}^{\alpha} R)(t) = \lambda R(t), R(0) = R_0 \in K_C(\mathbb{S}), t \in [0,1], \tag{16}$$

Where $\lambda \epsilon [-1,1]\{0\}$.

Scenario 1:

Consider $\lambda \in (0,1]$ and R be $\mathfrak{w}-$rising then the methodical resolution of calculation is provided by $R(t) = \left[\underline{R}(t), \overline{R}(t)\right] = R_0 E_\alpha[\lambda t^\alpha]$. Table 1 lists the discrepancies between the estimate and analytical results.

Scenario 2:

Consider $\lambda \in [-1,0)$ and R be $\mathfrak{w}-$falling then the methodical resolution of calculation is provided by $R(t) = [\underline{R}(t), \overline{R}(t)] = R_0 \mathrm{E}_\alpha[\lambda \mathrm{t}^\alpha]$. Table 2 lists the discrepancies between the analytical and estimate outcomes.

Table 1: Errors in Scenario 1

Step size$h$	$\underline{E}_{MFEM}$	$\overline{E}_{MFEM}$	$\underline{E}_{TM}$	$\overline{E}_{TM}$
1/10	0.2712	0.5824	0.1351	0.1446
1/20	0.1366	0.2631	0.063	0.0723
1/40	0.0653	0.1427	0.0331	0.0361
1/80	0.324	0.0648	0.0162	0.0181
1/160	0.0159	0.0319	8.1E-03	8.03E-03
1/320	0.0069	0.0142	4.03E-03	4.35E-03
1/640	0.0035	0.0070	2.08E-03	2.18E-03

Table 2: Errors in Scenario 2

Step size h	$\underline{E}_{MFEM}$	$\overline{E}_{MFEM}$	$\underline{E}_{TM}$	$\overline{E}_{TM}$
1/10	3.65E-02	6.50E-02	1.39E-02	2.25E-02
1/20	1.62E-02	3.28E-02	1.15E-02	1.39E-02
1/40	6.24E-03	1.41E-02	0.39E-02	0.59E-02

1/80	3.38E-03	5.80E-03	0.25E-02	0.38E-02
1/160	1.88E-03	3.86E-03	0.19E-02	0.35E-02
1/320	7.34E-04	1.58E-03	0.49E-03	0.21E-02
1/640	4.24E-04	7.35E-04	0.37E-03	0.52E-03

The table you provided shows the errors associated with different step sizes in numerical methods for solving ordinary differential equations (ODEs). The step size h is varied, and the errors for two methods, denoted as MFEM (likely a modified finite element method) and TM (Taylor method), are recorded

Step Size h

The step size h is the interval between points at which the numerical solution is evaluated. A smaller step size typically leads to a more accurate approximation but requires more computational effort.

References

[1] J. Larsson, "eulerr: Area-Proportional Euler Diagrams with Ellipses".

[2] Y. Luo *et al.*, "High-Order Taylor Expansion-Based Nonlinear Value Function Approximation for Stochastic Economic Dispatch of Active Distribution Network," *IEEE Trans. Smart Grid*, pp. 1–1, 2024, doi: 10.1109/TSG.2024.3395497.

[3] "Simulation optimization: a review of algorithms and applications | Annals of Operations Research." Accessed: Jul. 17, 2024. [Online]. Available: https://link.springer.com/article/10.1007/s10479-015-2019-x

[4] K. Pentland, "Towards probabilistic time-parallel algorithms for solving initial value problems," phd, University of Warwick, 2023. Accessed: Jul. 17, 2024. [Online]. Available: http://webcat.warwick.ac.uk/record=b3982103

[5] "Artificial neural networks: a practical review of applications involving fractional calculus | The European Physical Journal Special Topics." Accessed: Jul. 17, 2024. [Online]. Available: https://link.springer.com/article/10.1140/epjs/s11734-022-00455-3?utm_campaign=HSCR_JRNLS_AWA1_GL_MPAS_TRDMD&utm_term=null&utm_content=paid&utm_source=trendmd&utm_medium=cpc

[6] Z.-W. Fang, H.-W. Sun, and H. Wang, "A fast method for variable-order Caputo fractional derivative with applications to time-fractional diffusion equations," *Comput. Math. Appl.*, vol. 80, no. 5, pp. 1443–1458, Sep. 2020, doi: 10.1016/j.camwa.2020.07.009.

[7] "Analytic solution of homogeneous time-invariant fractional IVP | Advances in Continuous and Discrete Models." Accessed: Jul. 17, 2024. [Online]. Available: https://link.springer.com/article/10.1186/s13662-018-1601-3

[8] V. Tarasov, "On History of Mathematical Economics: Application of Fractional Calculus," *Mathematics*, vol. 7, no. 6, p. 509, Jun. 2019, doi: 10.3390/math7060509.

[9] S. Rezapour, M. S. Souid, Z. Bouazza, A. Hussain, and S. Etemad, "On the Fractional Variable Order Thermostat Model: Existence Theory on Cones via Piece-Wise Constant Functions," *J. Funct. Spaces*, vol. 2022, pp. 1–11, Mar. 2022, doi: 10.1155/2022/8053620.

[10] "Discriminative Embeddings of Latent Variable Models for Structured Data." Accessed: Jul. 17, 2024. [Online]. Available: https://proceedings.mlr.press/v48/daib16.html

[11] L. P. Castro and A. S. Silva, "On the Existence and Stability of Solutions for a Class of Fractional Riemann–Liouville Initial Value Problems," *Mathematics*, vol. 11, no. 2, p. 297, Jan. 2023, doi: 10.3390/math11020297.

[12] C. Derbazi, H. Hammouche, and Laboratory of Mathematics and Applied Sciences University of Ghardaia, 47000, Algeria, "Caputo-Hadamard fractional differential equations with nonlocal fractional integro-differential boundary conditions via topological degree theory," *AIMS Math.*, vol. 5, no. 3, pp. 2694–2709, 2020, doi: 10.3934/math.2020174.

[13] "Semi-implicit Hermite–Galerkin Spectral Method for Distributed-Order Fractional-in-Space Nonlinear Reaction–Diffusion Equations in Multidimensional Unbounded Domains | Journal of Scientific Computing." Accessed: Jul. 17, 2024. [Online]. Available: https://link.springer.com/article/10.1007/s10915-020-01320-y

[14] "Exact and numerical solutions of higher-order fractional partial differential equations: A new analytical method and some applications | Pramana." Accessed: Jul. 17, 2024. [Online]. Available: https://link.springer.com/article/10.1007/s12043-022-02446-4

[15] A. W. Bergersen, "Investigating the Link Between Patient-specific Morphology and Hemodynamics: Implications for Aneurism Initiation?," Master thesis, 2016. Accessed: Jul. 17, 2024. [Online]. Available: https://www.duo.uio.no/handle/10852/50515

[16] "Robust Time Spectral Methods for Solving Fractional Differential Equations in Finance - ProQuest." Accessed: Jul. 17, 2024. [Online]. Available: https://www.proquest.com/openview/052617e617c31f9bf8066fd63093bf10/1?pq-origsite=gscholar&cbl=2026366&diss=y

[17] G. Greco and F. Scarcello, "The Power of Local Consistency in Conjunctive Queries and Constraint Satisfaction Problems," *SIAM J. Comput.*, vol. 46, no. 3, pp. 1111–1145, Jan. 2017, doi: 10.1137/16M1090272.

[18] "Approximation Theory of Tree Tensor Networks: Tensorized Univariate Functions | Constructive Approximation." Accessed: Jul. 17, 2024. [Online]. Available: https://link.springer.com/article/10.1007/s00365-023-09620-w

[19] "Multi-interfaces Entity Model for Summative Risk Management of Medical Products | SpringerLink." Accessed: Jul. 17, 2024. [Online]. Available: https://link.springer.com/chapter/10.1007/978-3-030-94399-8_12